Asymptotic incidence energy and Laplacian-energy-like invariant of the Union Jack lattice

Jia-Bao Liu[a,b], Xiang-Feng Pan[a,*]

[a] School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China

[b] Department of Public Courses, Anhui Xinhua University, Hefei 230088, P. R. China

Abstract

The incidence energy $\mathscr{I}\mathscr{E}(G)$ of a graph G, defined as the sum of the singular values of the incidence matrix of a graph G, is a much studied quantity with well known applications in chemical physics. The Laplacian-energy-like invariant of G is defined as the sum of square roots of the Laplacian eigenvalues. In this paper, we obtain the closed-form formulae expressing the incidence energy and the Laplacian-energy-like invariant of the Union Jack lattice. Moreover, the explicit asymptotic values of these quantities are calculated by utilizing the applications of analysis approach with the help of calculational software.

Keywords: Solvable lattice models; Random graphs; Networks; Incidence energy; Laplacian-energy-like invariant

1 Introduction

Lattices have several attractive features that make them interesting candidates for use in matter physics. The quantum spin model with frustration and the Ising model of the Union Jack lattice have been exploited extensively by physicists [1].

The problem of calculation of some physical and chemical indices (such as the energy, the incidence energy and the Laplacian-energy-like invariant) on the lattices has been extensively studied and it became a popular topic of research in mathematical chemistry and mathematics. The energy of a graph G arising in chemical physics, is defined as the sum of the absolute values of the eigenvalues of G. The energy of many lattices were considered by physicists [1, 2, 4, 28]. A general problem of interest in physics, chemistry and mathematics is the calculations of various energies of lattices [1, 3, 4, 5, 27]. The closed-form formulae expressing the incidence energy of the hexagonal lattice, triangular lattice, and $3^3.4^2$ lattice are investigated in [29]. The Laplacian-energy-like invariant of hexagonal, triangular, and $3^3.4^2$ lattices with three boundary conditions are reported in [30].

The authors of [1] investigated the formulae of the number of spanning trees, the energy, and the Kirchhoff index of the Union Jack lattice with toroidal boundary condition. Although the incidence

*Corresponding author. Tel:+86-551-63861313. E-mail:liujiabaoad@163.com (J.Liu), xfpan@ahu.edu.cn(X.Pan).

energy and the Laplacian-energy-like invariant of some lattices are reported by mathematics and physics journals (see for example Refs. [29, 30]). But as far as we know, no one has considered the incidence energy and the Laplacian-energy-like invariant of the Union Jack lattice. In this paper, we obtain the solution of these problems.

The rest of the paper is organized as follows. We introduce the preliminaries and the definitions of lattices in Sections 2 and 3, respectively. In section 4, we deduce the signless Laplacian eigenvalues and the incidence energy of $UJL(n,m)$. In section 5, we investigate the Laplacian-energy-like invariant of the Union Jack lattice.

2 Preliminaries

At the beginning of this section, we first introduce some notations, which will be used in following discussion.

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote by $\mid V(G) \mid$ and $\mid E(G) \mid$ the numbers of vertices and edges, respectively. The adjacency matrix of graph G is an $n \times n$ $(0,1)$-matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if and only if (v_i, v_j) is an edge of G and $a_{ij} = 0$ otherwise. Let $\lambda_1(G) \geq \lambda_2(G) \geq \ldots \geq \lambda_n(G)$ be the eigenvalues of the adjacency matrix $A(G)$.

$$Spec_A(G) = \Big\{\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)\Big\}$$

is also called the spectrum of G [20].

The degree of a vertex v, denoted by $d_G(v)$, is the number of edges incident to v in a graph G. Let $D(G)$ be the diagonal matrix of vertex degrees of G. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the Laplacian spectrum of G is denoted by

$$Spec_L(G) = \Big\{\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\Big\},$$

where $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$ are the eigenvalues of the Laplacian matrix $L(G)$. The signless Laplacian matrix is $Q(G) = D(G) + A(G)$ and the signless Laplacian spectrum of G is denoted by

$$Spec_Q(G) = \Big\{q_1(G), q_2(G), \ldots, q_n(G)\Big\},$$

where $q_1(G), q_2(G), \ldots, q_n(G)$ are the eigenvalues of the signless Laplacian matrix $Q(G)$. It is well known that $L(G)$ and $Q(G)$ are symmetric and positive semi-definite, and the spectra of $L(G)$ and $Q(G)$ coincide if and only if the graph G is bipartite [21, 22].

Let $I(G)$ be the (vertex-edge) incidence matrix of the graph G. The (i,j)-entry of $I(G)$ is 1 if v_i is incident with e_j and 0 otherwise. (In what follows, the unit matrix of order n will be denoted by E_n to avoid confusion with the incidence matrix.) A well known fact is the identity [20]:

$$I(G)I(G)^t = A(G) + D(G) = Q(G).$$

We recall Cartesian product of graphs and Kronecker product, which will be used in the proof of our result.

Given graphs G and H with vertex sets U and V, the Cartesian product $G\Box H$ of graphs G and H is a graph such that the vertex set of $G\Box H$ is the Cartesian product $U\Box V$; and any two vertices (u,u') and (v,v') are adjacent in $G\Box H$ if and only if either $u=v$ and u' is adjacent v' in H, or $u'=v'$ and u is adjacent v in G [23].

The Kronecker product $A\otimes B$ of two matrices $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{p\times q}$ is the $mp\times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. If A,B,C and D are matrices of such size that one can form the matrix products AC and BD, then $(A\otimes B)(C\otimes D)=AC\otimes BD$. It follows that $A\otimes B$ is invertible if and only if A and B are invertible, in which case the inverse is given by $(A\otimes B)^{-1}=A^{-1}\otimes B^{-1}$. Note also that $(A\otimes B)^T=A^T\otimes B^T$. Moreover, if the matrices A and B are of order $n\times n$ and $p\times p$, respectively, then $det(A\otimes B)=(det\ A)^p(det\ B)^n$. The readers are referred to [24] for other properties of the Kronecker product not mentioned here.

3 The 4.8.8 lattice and the Union Jack lattice

Our notations for the 4.8.8 lattice and the Union Jack lattice follow [1]. The 4.8.8 lattice with toroidal boundary conditions, denoted by $G^t(n,m)$, is illustrated in Figure 1 (a). The left and right (resp. the lower and upper) boundaries of the picture are identified such that all a_is, a_i^*s, b_is, b_i^*s, are some vertices on the left, right, lower and upper boundaries, respectively, and $(a_1,a_1^*),(a_2,a_2^*),\ldots,(a_m,a_m^*)$ and $(b_1,b_1^*),(b_2,b_2^*),\ldots,(b_n,b_n^*)$ are edges in $G^t(n,m)$. Obviously, the 4.8.8 lattice $G^t(n,m)$ is composed of mn quadrangles.

The Union Jack lattice with toroidal boundary condition, denoted by $UJL(n,m)$, is the dual lattice of the 4.8.8 lattice with toroidal boundary condition. Figure 1 (b) is the Union Jack lattice $UJL(n,m)$ corresponding to the 4.8.8 lattice illustrated in Figure 1 (a). Similarly, the left and right (resp. the lower and upper) boundaries of the picture are identified such that all c_is, c_i^*s, d_is, d_i^*s, are some vertices on the left, right, lower and upper boundaries, respectively, and $(c_1,c_1^*),(c_2,c_2^*),\ldots,(c_m,c_m^*)$ and $(d_1,d_1^*),(d_2,d_2^*),\ldots,(d_n,d_n^*)$ are edges in $UJL(n,m)$.

The Union Jack lattice with toroidal boundary condition $UJL(n,m)$ also can be obtained from an $n\times m$ square lattice with toroidal boundary condition by inserting a new vertex v_f to each face f and adding four edges $(v_f,u_i(f)),i=1,2,\ldots,4$, where $u_i(f)$ are four vertices on the boundary of f. Let G' be the graph obtained from $UJL(n,m)$ by deleting the vertices of degree 4, i.e., G' is an $n\times m$ square lattice with toroidal boundary condition, which is the Cartesian product $C_n\Box C_m$ of two cycles C_n and C_m. It is easy to see that $UJL(n,m)$ has $2mn$ vertices.

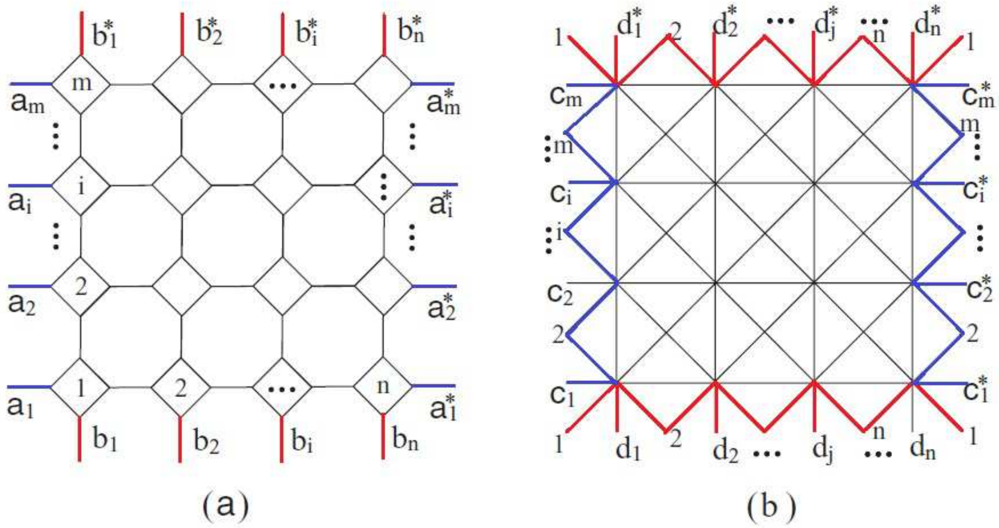

Figure 1: (a) The 4.8.8 lattice $G^t(n,m)$ with toroidal boundary condition. (b) The Union Jack lattice $UJL(n,m)$ with toroidal boundary condition, i.e., the dual graph of the 4.8.8 lattice $G^t(n,m)$.

4 The signless Laplacian eigenvalues and the incidence energy of $UJL(n,m)$

4.1 The signless Laplacian eigenvalues of $UJL(n,m)$.

In this subsection, we will deduce the signless Laplacian eigenvalues of $UJL(n,m)$ by utilizing the techniques in [1]. For the convenience to following description, we set

$$\alpha_i = \frac{2\pi i}{n}, \beta_j = \frac{2\pi j}{m}, i = 0, 1, \ldots, n-1; j = 0, 1, \ldots, m-1.$$

Theorem 4.1 *Let $UJL(n,m)$ be a Union Jack lattice with toroidal boundary condition. Then the signless Laplacian eigenvalues of $UJL(n,m)$ are*

$$(6 + \cos\alpha_i + \cos\beta_j) \pm \sqrt{(6 + \cos\alpha_i + \cos\beta_j)^2 - 4(7 + \cos\alpha_i + \cos\beta_j - cos\alpha_i \cos\beta_j)},$$

where $0 \le i \le n-1, 0 \le j \le m-1$.

Proof. With a suitable labelling of vertices of $UJL(n,m)$, the adjacency matrix of $UJL(n,m)$ is

$$A(UJL(n,m)) = \begin{pmatrix} A(C_n \Box C_m) & M \\ M^T & 0 \end{pmatrix},$$

where $A(C_n\square C_m)$ is the adjacency matrix of $C_n\square C_m$, M is the matrix induced by the adjacency relation between $V(C_n\square C_m)$ and $V(F)$, here F is the face set of $C_n\square C_m$, M^T is the transpose of M. Moreover, M is a block matrix (with m times m entries) which has the following form:

$$M=\begin{pmatrix} R & 0 & 0 & \dots & 0 & R \\ R & R & 0 & \dots & 0 & 0 \\ 0 & R & R & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & R & 0 \\ 0 & 0 & 0 & \dots & R & R \end{pmatrix}_{m\times m}, \text{ where } R=\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}_{n\times n}.$$

On one hand, the degrees of $UJL(n,m)$ are 4 or 8. With a suitable labelling of vertices of $UJL(n,m)$, the diagonal matrix $D(UJL(n,m))$ is

$$D(UJL(n,m)) = diag\{\underbrace{4,4,\dots,4}_{mn},\underbrace{8,8,\dots,8}_{mn}\}. \tag{1}$$

On the other hand, the adjacency matrix $A(C_n\square C_m)$ of $C_n\square C_m$ has the following form by a suitable labelling of vertices of $C_n\square C_m$:

$$\begin{aligned} A\Big(C_n\square C_m\Big) &= \begin{pmatrix} A(C_n) & E_n & 0 & \dots & 0 & E_n \\ E_n & A(C_n) & E_n & \dots & 0 & 0 \\ 0 & E_n & A(C_n) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & A(C_n) & E_n \\ E_n & 0 & 0 & \dots & E_n & A(C_n) \end{pmatrix}_{n\times n} \\ &= E_m\otimes A(C_n) + A(C_m)\otimes E_n, \end{aligned} \tag{2}$$

where E_n is the identity matrix of order n.

Based on equations (1) and (2), the signless Laplacian matrix $Q(UJL(n,m))$ has the following form:

$$\begin{aligned} Q(UJL(n,m)) &= D(UJL(n,m)) + A(UJL(n,m)) \\ &= \begin{pmatrix} 8E_{mn}+A(C_n\square C_m) & M \\ M^T & 4E_{mn} \end{pmatrix}, \end{aligned}$$

Hence the signless Laplacian characteristic polynomial of $UJL(n,m)$ is

$$\begin{aligned} \psi(UJL(n,m),x) &= det\ (xE_{2mn} - Q(UJL(n,m)) \\ &= det \begin{pmatrix} (x-8)E_{mn}-A(C_n\square C_m) & -M \\ -M^T & (x-4)E_{mn} \end{pmatrix} \\ &= det\Big((x-4)(x-8)E_{mn}-(x-4)A(C_n\square C_m)-MM^T\Big). \end{aligned} \tag{3}$$

Considering that

$$
\begin{aligned}
MM^T &= \begin{pmatrix}
2RR^T & RR^T & 0 & \dots & 0 & RR^T \\
RR^T & 2RR^T & RR^T & \dots & 0 & 0 \\
0 & RR^T & 2RR^T & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 2RR^T & RR^T \\
RR^T & 0 & 0 & \dots & RR^T & 2RR^T
\end{pmatrix} \\
&= B_m \otimes B_n.
\end{aligned}
$$

where

$$
B_n = \begin{pmatrix}
2 & 1 & 0 & \dots & 0 & 1 \\
1 & 2 & 1 & \dots & 0 & 0 \\
0 & 1 & 2 & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \dots & 2 & 1 \\
1 & 0 & 0 & \dots & 1 & 2
\end{pmatrix}_{n\times n} = 2E_n + A(C_n).
$$

Consequently,

$$
MM^T = \Big(2E_m + A(C_m)\Big) \otimes \Big(2E_n + A(C_n)\Big). \tag{4}
$$

According to equations (2), (3) and (4), we have

$$
\begin{aligned}
\psi(UJL(n,m),x) &= det\Big((x-4)(x-8)E_{mn} - (x-4)A(C_n\Box C_m) - MM^T\Big) \\
&= det\Bigg\{(x-4)(x-8)E_{mn} - (x-4)\Big[E_m \otimes A(C_n) + A(C_m) \otimes E_n\Big] \\
&\quad -\Big(2E_m + A(C_m)\Big) \otimes \Big(2E_n + A(C_n)\Big)\Bigg\} \\
&= det\Bigg\{(x^2-12x+28)E_{mn} - (x-2)\Big[E_m \otimes A(C_n) + A(C_m) \otimes E_n\Big] \\
&\quad -A(C_m) \otimes A(C_n)\Bigg\}.
\end{aligned}
$$

Let

$$
Z = (x^2-12x+28)E_{mn} - (x-2)[E_m \otimes A(C_n) + A(C_m) \otimes E_n] - A(C_m) \otimes A(C_n). \tag{5}
$$

Note that the eigenvalues of $A(C_n)$ are $2\cos\alpha_i, 0 \le i \le n-1$.

Hence, there exist two invertible matrices P and Q such that

$$
P^{-1}A(C_n)P = diag\Big(2, 2\cos\alpha_i, \dots, 2\cos\alpha_{n-1}\Big) := W_n,
$$

and

$$Q^{-1}A(C_m)Q = diag\Big(2, 2\cos\alpha_i, \ldots, 2\cos\alpha_{m-1}\Big) := W_m.$$

In fact, P and Q are invertible matrices, then $Q \otimes P$ is an invertible matrix.

By the equation (5), one can obtain

$$\begin{aligned}(Q\otimes P)^{-1}Z(Q\otimes P) &= (Q\otimes P)^{-1}\Big\{(x^2-12x+28)E_{mn}-(x-2)\Big[E_m\otimes A(C_n)\\ &\quad +A(C_m)\otimes E_n\Big]-A(C_m)\otimes A(C_n)\Big\}(Q\otimes P)\\ &= (x^2-12x+28)E_{mn}-(x-2)\Big[E_m\otimes W_n+W_m\otimes E_n\Big]-W_m\otimes W_n.\end{aligned}$$

It is not difficult to see that $(x^2-12x+28)E_{mn}-(x-2)\Big[E_m\otimes W_n+W_m\otimes E_n\Big]-W_m\otimes W_n$ is a diagonal matrix whose diagonal entries are $(x^2-12x+28)-(x-2)(2\cos\alpha_i+2\cos\beta_j)-4\cos\alpha_i\cos\beta_j$.

Actually, the zeros of $(x^2-12x+28)-(x-2)(2\cos\alpha_i+2\cos\beta_j)-4\cos\alpha_i\cos\beta_j=0$ are

$$(6+\cos\alpha_i+\cos\beta_j)\pm\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}.$$

Then the proof of this theorem is complete. ∎

4.2 The incidence energy of $UJL(n,m)$

Gutman introduced the concept of energy $\mathscr{E}(G)$ [6] for a simple graph G, which is defined as

$$\mathscr{E}(G)=\sum_{i=1}^{n}|\lambda_i(G)|.$$

As an analogue to $\mathscr{E}(G)$, the incidence energy $\mathscr{I}\mathscr{E}(G)$, is a novel topological index. Inspired by Nikiforov's idea [7], in 2009 Jooyandeh et al. [8] introduced the concept of incidence energy $\mathscr{I}\mathscr{E}(G)$ of a graph G, defining it as the sum of the singular values of the incidence matrix $I(G)$, i.e.,

$$\mathscr{I}\mathscr{E}(G)=\sum_{i=1}^{n}\sigma_i,$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ are the singular values of the incidence matrix $I(G)$.

Since the identity [20]: $I(G)I(G)^t=A(G)+D(G)=Q(G)$, its immediate consequence is

$$\sigma_i=\sqrt{q_i}.$$

Therefore

$$\mathscr{I}\mathscr{E}(G)=\sum_{i=1}^{n}\sqrt{q_i}.$$

For more work on $\mathscr{I}\mathscr{E}(G)$, the readers are referred to [9, 10, 11] and recent articles [12, 13, 14, 15, 16, 17, 18].

Next, we will explore the incidence energy $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$ of $UJL(n,m)$.

Theorem 4.2 *Let $UJL(n,m)$ be a Union Jack lattice with toroidal boundary condition. Then*

1. *The incidence energy $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$ of $UJL(n,m)$ can be expressed by*

$$
\begin{aligned}
&\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)\\
= &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6+\cos\alpha_i+\cos\beta_j)+\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}}\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6+\cos\alpha_i+\cos\beta_j)-\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}},
\end{aligned}
$$

where $\alpha_i=\frac{2\pi i}{n},\beta_j=\frac{2\pi j}{m},i=0,1,\ldots,n-1;j=0,1,\ldots,m-1.$

2. *As $m,n\to\infty$, $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)\approx 9.4770mn.$*

Proof. By Theorem 4.1 and the definition of the incidence energy $\mathscr{I}\mathscr{E}(G)$, one can obtain the incidence energy $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$ of $UJL(n,m)$ is

$$
\begin{aligned}
&\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)\\
= &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6+\cos\alpha_i+\cos\beta_j)+\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}}\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6+\cos\alpha_i+\cos\beta_j)-\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}}.
\end{aligned}
$$

Therefore the statement 1 of Theorem 4.2 is immediate.

In what follows, we will calculate the asymptotic value of the incidence energy $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$. For the sake of simplicity , we set

$$A=(6+\cos\alpha_i+\cos\beta_j)+\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)},$$

$$B=(6+\cos\alpha_i+\cos\beta_j)-\sqrt{(6+\cos\alpha_i+\cos\beta_j)^2-4(7+\cos\alpha_i+\cos\beta_j-cos\alpha_i\cos\beta_j)}.$$

Considering that m,n approach infinity, we have

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)}{2mn}$$

$$= \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{A}\cdot\ dxdy + \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{B}\cdot\ dxdy \tag{6}$$
$$\approx 4.7385. \tag{7}$$

Consequently, according to the equality (7), we can get the asymptotic incidence energy $\mathscr{IE}\Big(UJL(n,m)\Big) \approx 9.4770mn$, as $m,n\to\infty$. ∎

Remark 4.3 The numerical integration value in equality (6) is calculated with MATLAB software calculation, i.e.,

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{A}\cdot\ dxdy \approx 2.9040, \quad \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{B}\cdot\ dxdy \approx 1.8345.$$

5 The Laplacian-energy-like invariant of the Union Jack lattice

The Laplacian-energy-like invariant of a graph G, $\mathscr{LEL}(G)$ for short, is defined as

$$\mathscr{LEL}(G) = \sum_{i=1}^{n-1}\sqrt{\mu_i},$$

which is a novel topological index. The concept of $\mathscr{LEL}(G)$ was first introduced by J. Liu and B. Liu ([19], 2008), where it showed that $\mathscr{LEL}(G)$ has similar features as the graph energy $\mathscr{E}(G)$ [26].

We recall the Laplacian eigenvalues of $UJL(n,m)$. The authors of [1] proved the following theorem, whose proof thus is omitted.

Theorem 5.1 *([1]) Let $UJL(n,m)$ be a Union Jack lattice with toroidal boundary condition. Then the Laplacian eigenvalues of $UJL(n,m)$ are*

$$(6-\cos\alpha_i-\cos\beta_j) \pm \sqrt{(6-\cos\alpha_i-\cos\beta_j)^2 - 4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)},$$

where $0\le i\le n-1, 0\le j\le m-1$.

The following theorem expresses the Laplacian-energy-like invariant of the Union Jack lattice.

Theorem 5.2 *Let $UJL(n,m)$ be a Union Jack lattice with toroidal boundary condition. Then*

1. *The Laplacian-energy-like invariant $\mathscr{LEL}\Big(UJL(n,m)\Big)$ of $UJL(n,m)$ can be expressed by*

$$\begin{aligned} &\mathscr{LEL}\Big(UJL(n,m)\Big) \\ =\ &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6-\cos\alpha_i-\cos\beta_j)+\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)}} \\ &+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6-\cos\alpha_i-\cos\beta_j)-\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)}}, \end{aligned}$$

where $\alpha_i = \frac{2\pi i}{n}, \beta_j = \frac{2\pi j}{m}, i = 0, 1, \ldots, n-1; j = 0, 1, \ldots, m-1.$

2. As $m, n \to \infty$, $\mathscr{LEL}\Big(UJL(n,m)\Big) \approx 9.3682mn$.

Proof. By Theorem 5.1 and the definition of the Laplacian-energy-like invariant $\mathscr{LEL}(G)$, one can obtain the $\mathscr{LEL}\Big(UJL(n,m)\Big)$ of $UJL(n,m)$ is

$$\begin{aligned}
&\mathscr{LEL}\Big(UJL(n,m)\Big)\\
=\ &\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6-\cos\alpha_i-\cos\beta_j)+\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)}}\\
&+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\sqrt{(6-\cos\alpha_i-\cos\beta_j)-\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)}}.
\end{aligned}$$

Hence the statement 1 of Theorem 5.2 holds.

Similarly, we will formulate the asymptotic value of the $\mathscr{LEL}\Big(UJL(n,m)\Big)$. Let

$$C = (6-\cos\alpha_i-\cos\beta_j)+\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)},$$

$$D = (6-\cos\alpha_i-\cos\beta_j)-\sqrt{(6-\cos\alpha_i-\cos\beta_j)^2-4(7-3\cos\alpha_i-3\cos\beta_j-cos\alpha_i\cos\beta_j)}.$$

Considering that m, n approach infinity, we have

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{\mathscr{LEL}\Big(UJL(n,m)\Big)}{2mn}\\
=\ &\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{C}\cdot\ dxdy+\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{D}\cdot\ dxdy \qquad (8)\\
\approx\ &4.6841. \qquad (9)
\end{aligned}$$

Consequently, according to the equality (9), we can get the asymptotic Laplacian-energy-like invariant $\mathscr{LEL}\Big(UJL(n,m)\Big) \approx 9.3682mn$, as $m, n \to \infty$.

Summing up, we complete the proof. □

Remark 5.3 The numerical integration value in equality (8) is calculated with MATLAB software calculation, i.e.,

$$\frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{C}\cdot\ dxdy \approx 2.9874, \quad \frac{1}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{D}\cdot\ dxdy \approx 1.6967.$$

Acknowledgments

The work of J. B. Liu is partly supported by the Natural Science Foundation of Anhui Province of China under Grant No. KJ2013B105 and the National Science Foundation of China under Grant Nos.11471016 and 11401004. The work of X. F. Pan is partly supported by the National Science Foundation of China under Grant Nos. 10901001, 11171097, and 11371028.

References

[1] S. Li, W. G. Yan, T. Tian, Some physical and chemical indices of the Union Jack lattice, J. Stat. Mech. P10004 (2015) 1-14.

[2] X. Y. Liu, W. G. Yan, The triangular kagomé lattices revisited, Physica A 392 (2013) 5615-5621.

[3] W. G. Yan, Z. H. Zhang, Asymptotic energy of lattices, Physica A 388 (2009) 1463-1471.

[4] L. Z. Ye, On the Kirchhoff index of some toroidal lattices, Linear Multilinear A. 59 (6) (2011) 645-650.

[5] W. Yan, L. Ye, On the minimal energy of trees with a given diameter, Appl. Math. Lett. 18 (2005) 1046-1052.

[6] X. Li, Y. Shi, I. Gutman, Graph Energy, Springer, New York, 2012.

[7] V. Nikiforov, The energy of graphs and matrices, J. Math. Anal. Appl. 326 (2007) 1472-1475.

[8] M. R. Jooyandeh, D. Kiani, M. Mirzakhah, Incidence energy of a graph, MATCH Commun. Math. Comput. Chem. 62 (2009) 561-572.

[9] I. Gutman, D. Kiani, M. Mirzakhah, On incidence energy of graphs, MATCH Commun. Math. Comput. Chem. 62 (2009)573-580.

[10] I. Gutman, D. Kiani, M. Mirzakhah, B. Zhou, On incidence energy of a graph, Linear Algebra and its Applications, 431 (2009)1223-1233.

[11] K. C. Das, I. Gutman, On incidence energy of graphs, Linear Algebra and its Applications, 446 (2014) 329-344.

[12] Ş. B. Bozkurt, D. Bozkurt, On incidence energy, MATCH Commun. Math. Comput. Chem. 72 (2014) 215-225.

[13] Ş. B. Bozkurt, I. Gutman, Estimating the incidence energy, MATCH Commun. Math. Comput. Chem.70 (2013) 143-156.

[14] O. Rojo, E. Lenes, A sharp upper bound on the incidence energy of graphs in terms of connectivity, Lin. Algebra Appl. 438 (2013) 1485-1493.

[15] Z. Tang, Y. Hou, On incidence energy of trees, MATCH Commun. Math. Comput. Chem. 66 (2011) 977-984.

[16] J. Zhang, J. Li, New results on the incidence energy of graphs, MATCH Commun. Math. Comput. Chem. 68 (2012) 777-803.

[17] J. Zhang, X. D. Zhang, The signless Laplacian coefficients and incidence energy of bicyclic graphs, Lin. Algebra Appl.439(2013) 3859-3869.

[18] J. B. Liu, J. Cao, J. Xie, On the incidence energy of some toroidal lattices, Abstract and Applied Analysis, ID:568153 (2014) 1-6.

[19] J. Liu, B. Liu, A Laplacian-energy-like invariant of a graph, MATCH Commun. Math. Comput. Chem. 59 (2008) 397-419.

[20] D. Cvetković, M. Doob, H. Sachs, Spectra of Graphs–Theory and Applications, Barth Verlag, Heidelberg, 1995.

[21] D. Cvetković, P. Rowlinson, S.K. Simić, Signless Laplacians of finite graphs, Linear Algebra Appl. 423 (1) (2007) 155-171.

[22] D. Cvetković D, S. K. Simić, Towards a spectral theory of graphs based on the signless Laplacian, I. Publ. Inst. Math.(Beograd)(NS), 2009, 85(99): 19-33.

[23] G. Chartrand, P. Zhang, Introduction to Graph Theory. McGraw-Hill, Kalamazoo, MI. 2004.

[24] R. A. Horn, C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.

[25] P. Hansen, C. Lucas, Bounds and conjectures for the signless Laplacian index of graphs, Linear Algebra and its Applications 432 (2010) 3319-3336.

[26] I. Gutman, B. Zhou, B. Furtula, The Laplacian-energy like invariant is an energy like invariant, MATCH Commun. Math. Comput. Chem. 64 (2010) 85-96.

[27] Z. Zhang, Some physical and chemical indices of clique-inserted lattices, Journal of Statistical Mechanics: Theory and Experiment, 10 (2013) P10004.

[28] J. B. Liu, X. F. Pan, J. Cao, F. F. Hu, A note on some physical and chemical indices of clique-inserted lattices, Journal of Statistical Mechanics: Theory and Experiment, 6 (2014), P06006.

[29] J. B. Liu, X. F. Pan, Asymptotic incidence energy of lattices, Physica A: Statistical Mechanics and its Applications, 422 (2015) 193-202.

[30] J. B. Liu, X. F. Pan, F. T. Hu, F. F. Hu, Asymptotic Laplacian-energy-like invariant of lattices, Appl. Math. Comput. 253 (2015) 205-214.

Highlights

- We deduce the signless Laplacian eigenvalues of $UJL(n,m)$.
- The formulae expressing $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$ and $\mathscr{L}\mathscr{E}\mathscr{L}\Big(UJL(n,m)\Big)$ are obtained.
- The explicit asymptotic values of $\mathscr{I}\mathscr{E}\Big(UJL(n,m)\Big)$ and $\mathscr{L}\mathscr{E}\mathscr{L}\Big(UJL(n,m)\Big)$ are calculated.